

# OPTIMAL LOWER BOUND OF THE RESONANCE WIDTHS FOR THE HELMHOLTZ RESONATOR

ANDRÉ MARTINEZ & LAURENCE NÉDÉLEC

**ABSTRACT.** Under a geometric assumption on the region near the end of its neck, we prove an optimal exponential lower bound on the widths of resonances for a general two-dimensional Helmholtz resonator. An extension of the result to the  $n$ -dimensional case,  $n \leq 12$ , is also obtained.

## 1. INTRODUCTION

A resonator consists of a bounded cavity (the chamber) connected to the exterior by a thin tube (the neck of the chamber). The frequencies of the sounds it produces are determined by the shape of the chamber, while their duration by the length and the width of the neck in a non-obvious way, and our goal is to understand these. Mathematically, this phenomenon is described by the resonances of the Dirichlet Laplacian  $-\Delta_\Omega$  on the domain  $\Omega$  consisting of the union of the chamber, the neck and the exterior (see Figure 1).

This article extends our previous work [MN], in that we are now able to handle regions where the shape of the exterior is quite general, although the shape of the neck stays the same. The main changes appear in sections 4, 5 and 6, where Carleman estimates are used, and Green's identity is replaced by an estimate to obtain a lower bound on the imaginary part of the resonances.

We recall that resonances are the eigenvalues of a complex deformation of  $-\Delta_\Omega$ ; their real and imaginary parts are the frequencies and inverses of the half-lives, respectively, of the corresponding vibrational modes. It is of obvious physical interest to estimate these two quantities as precisely as possible. One practical way to do this involves studying this problem in the asymptotic limit when the width  $\varepsilon$  of the neck tends to zero. Those resonances with imaginary parts tending to zero converge to the eigenvalues

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<sup>1</sup>A.M.: Università di Bologna, Dipartimento di Matematica, Piazza di Porta San Donato 5, 40127 Bologna, Italy. Partly supported by Università di Bologna, Funds for Selected Research Topics and Funds for Agreements with Foreign Universities

L.N.: Department of Mathematics, Stanford University, Stanford, California 94305

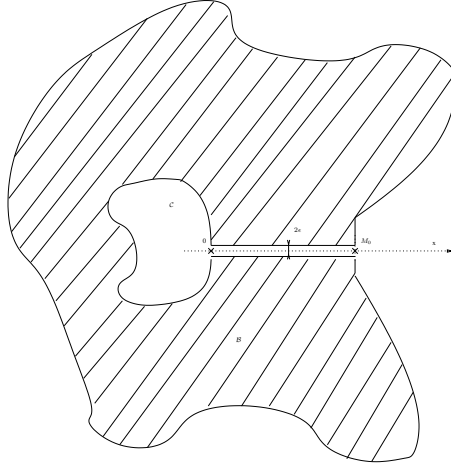


FIGURE 1. The Helmholtz resonator

of the Dirichlet Laplacian on the cavity, and there is an exponentially small upper bound for the absolute values of the imaginary parts (the widths) of the resonances [HM]. However, without very restrictive hypotheses, no lower bound is known. We mention in particular that lower bounds are known in the one-dimensional case [Ha, HaSi]. As for the higher dimensional case, we mention [FL, Bu2, HS] which contain results concerning exponentially small widths of quantum resonances, but these do not apply to a Helmholtz resonator. We also mention that the semiclassical lower bound obtained in [HS] is optimal (see also [FLM] for a generalization).

Here, we obtain an optimal lower bound (see Theorem 2.1) under a geometric condition concerning the external end part of the neck. Namely, we assume that the neck meets the boundary of the external region perpendicularly to it, and that the boundary is flat there (see (2.1) and Figure 1). This assumption is probably purely technical and should not be necessary. However, it permits us to adapt to this case some of the arguments of [MN], in order to obtain the lower bound after reducing the problem to an estimate near the end part of the neck. This reduction itself is obtained using Carleman estimates up to the boundary, as in [LL, LR].

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## 2. GEOMETRICAL DESCRIPTION AND RESULTS

Consider a Helmholtz resonator in  $\mathbb{R}^2$  consisting of a regular bounded open set  $\mathcal{C}$  (the cavity), connected to a regular unbounded open exterior domain  $\mathbf{E}$  through a thin straight tube  $\mathcal{T}(\varepsilon)$  (the neck) of radius  $\varepsilon > 0$  (see figure 2). We shall suppose that  $\varepsilon$  is very small.

To state this more precisely, let  $\mathcal{C}$  and  $\mathcal{B}$  be two bounded domains in  $\mathbb{R}^2$  with  $\mathcal{C}^\infty$  boundary; their closures and boundaries are denoted  $\overline{\mathcal{C}}$ ,  $\overline{\mathcal{B}}$  and  $\partial\mathcal{C}$ ,  $\partial\mathcal{B}$ . We assume that Euclidean coordinates  $(x, y)$  can be chosen in such a way that, for some  $L, \varepsilon_0 > 0$ , one has,

$$(2.1) \quad \begin{aligned} &\overline{\mathcal{C}} \subset \mathcal{B}; \quad (0, 0) \in \partial\mathcal{C}; (L, 0) \in \partial\mathcal{B}; \\ &[0, L] \times \{0\} \subset \overline{\mathcal{B}} \setminus \mathcal{C}; \quad \{L\} \times [-\varepsilon_0, \varepsilon_0] \subset \partial\mathcal{B}. \end{aligned}$$

Setting  $\mathcal{T}(\varepsilon) := [-\varepsilon_0, L] \times (-\varepsilon, \varepsilon) \cap (\mathbb{R}^2 \setminus \mathcal{C})$ ,  $\mathcal{C}(\varepsilon) = \mathcal{C} \cup \mathcal{T}(\varepsilon)$  and  $\mathbf{E} := \mathbb{R}^2 \setminus \overline{\mathcal{B}}$ , then the resonator is defined as,

$$\Omega(\varepsilon) := \mathcal{C}(\varepsilon) \cup \mathbf{E}.$$

As  $\varepsilon \rightarrow 0^+$ , the resonator  $\Omega(\varepsilon)$  collapses to  $\Omega_0 := \mathcal{C} \cup [0, M_0] \cup \mathbf{E}$ , where  $M_0$  is the point  $(L, 0) \in \mathbb{R}^2$ .

For any domain  $Q$ , let  $P_Q$  denote the Laplacian  $-\Delta_Q$  with Dirichlet boundary conditions on  $\partial Q$ ; for brevity, we write  $P_{\Omega_\varepsilon}$  as  $P_\varepsilon$ .

The resonances of  $P_\varepsilon$  are defined as the eigenvalues of the operator obtained by performing a complex dilation with respect to the coordinates  $(x, y)$ , for  $|x| + |y|$  large. We are interested in those resonances of  $P_\varepsilon$  that are close to the eigenvalues of  $P_{\mathcal{C}}$ . Thus let  $\lambda_0 > 0$  be an eigenvalue of  $P_{\mathcal{C}}$  with  $u_0$  the corresponding (normalized) eigenfunction. We make the following

Assumption **(H)**:

- $\lambda_0$  is simple;
- $u_0$  does not vanish on  $\mathcal{C}$  near the point  $(0, 0)$ .

Note that these properties are automatically satisfied when  $\lambda_0$  is the lowest eigenvalue of  $-\Delta_{\mathcal{C}}$ . When  $\lambda_0$  is a higher eigenvalue, then the last property means that 0 does not lie on the closure of a nodal line of  $u_0$ .

By the arguments of [HM], we know that there is a resonance  $\rho(\varepsilon) \in \mathbb{C}$  of  $P_\varepsilon$  such that  $\rho(\varepsilon) \rightarrow \lambda_0$  as  $\varepsilon \rightarrow 0$ . Furthermore, there is an eigenvalue  $\lambda(\varepsilon)$  of  $P_{\mathcal{C}(\varepsilon)}$  such that, for any  $\delta > 0$ ,

$$(2.2) \quad |\rho(\varepsilon) - \lambda(\varepsilon)| \leq C_\delta e^{-\pi(1-\delta)L/\varepsilon},$$

for some  $C_\delta > 0$  and all sufficiently small  $\varepsilon > 0$ . In particular, since  $\lambda(\varepsilon) \in \mathbb{R}$ , this gives

$$(2.3) \quad |\operatorname{Im} \rho(\varepsilon)| \leq C_\delta e^{-\pi(1-\delta)L/\varepsilon}.$$

We now state our main result.

**Theorem 2.1.** *Under Assumption (H), for any  $\delta > 0$  there exists  $C_\delta > 0$  such that, for all  $\varepsilon > 0$  small enough, one has*

$$|\operatorname{Im} \rho(\varepsilon)| \geq \frac{1}{C_\delta} e^{-\pi(1+\delta)L/\varepsilon}.$$

**Remark 2.2.** *We extend this result to the higher dimensional case in Section 11.*

**Remark 2.3.** *Gathering (2.3) and Theorem 2.1, we can reformulate the result as :*

$$(2.4) \quad \lim_{\varepsilon \rightarrow 0_+} \varepsilon \ln |\operatorname{Im} \rho(\varepsilon)| = -\pi L.$$

### 3. PROPERTIES OF THE RESONANT STATE

By definition, the resonance  $\rho(\varepsilon)$  is an eigenvalue of the complex distorted operator,

$$P_\varepsilon(\mu) := U_\mu P_\varepsilon U_\mu^{-1},$$

where  $\mu > 0$  is a small parameter, and  $U_\mu$  is a complex distortion of the form,

$$U_\mu \varphi(x, y) := \varphi((x, y) + i\mu f(x, y)),$$

with  $f \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ ,  $f = 0$  near  $\bar{\mathcal{B}}$ ,  $f(x, y) = (x, y)$  for  $|(x, y)|$  large enough. (Observe that by Weyl Perturbation Theorem, the essential spectrum of  $P_\varepsilon(\mu)$  is  $e^{-2i\alpha}\mathbb{R}_+$ , with  $\alpha = \arctan \mu$ .)

It is well known that such eigenvalues do not depend on  $\mu$  (see, e.g., [SZ, HeM]), and that the corresponding eigenfunctions are of the form  $U_\mu u_\varepsilon$  with  $u_\varepsilon$  independent of  $\mu$ , smooth on  $\mathbb{R}^2$  and analytic in a complex sector around  $\mathbf{E}$ . In other words,  $u_\varepsilon$  is a non trivial analytic solution of the equation  $-\Delta u_\varepsilon = \rho(\varepsilon)u_\varepsilon$  in  $\Omega(\varepsilon)$ , such that  $u_\varepsilon|_{\partial\Omega(\varepsilon)} = 0$  and, for all  $\mu > 0$  small enough,  $U_\mu u_\varepsilon$  is well defined and is in  $L^2(\Omega(\varepsilon))$  (in our context, this latter property will be taken as a definition of the fact that  $u_\varepsilon$  is *outgoing*). Moreover,  $u_\varepsilon$  can be normalized by setting, for some fixed  $\mu > 0$ ,

$$\|U_\mu u_\varepsilon\|_{L^2(\Omega(\varepsilon))} = 1.$$

In that case, we learn from [HM] (in particular Proposition 3.1 and formula (5.13)), that, for any  $\delta > 0$ , and for any  $R > 0$  large enough, one has,

$$(3.1) \quad \|u_\varepsilon\|_{L^2(\Omega(\varepsilon) \cap \{|(x,y)| < R\})} \geq 1 - \mathcal{O}(e^{(\delta - \frac{\pi L}{2})/\varepsilon}),$$

and

$$(3.2) \quad \|u_\varepsilon\|_{H^1(\mathbf{E} \cap \{|(x,y)| < R\})} = \mathcal{O}(e^{(\delta - \frac{\pi L}{2})/\varepsilon}).$$

Now, we take  $R > 0$  such that  $\bar{\mathcal{B}} \subset \{|(x, y)| < R\}$ . Using the equation  $-\Delta u_\varepsilon = \rho u_\varepsilon$  and Green's formula on the domain  $\Omega(\varepsilon) \cap \{|(x, y)| < R\}$ , and using polar coordinates  $(r, \theta)$ , we obtain,

$$\operatorname{Im} \rho \int_{\Omega(\varepsilon) \cap \{|(x,y)| < R\}} |u_\varepsilon|^2 dx dy = - \operatorname{Im} \int_0^{2\pi} \frac{\partial u_\varepsilon}{\partial r}(R, \theta) \bar{u}_\varepsilon(R, \theta) R d\theta,$$

and thus, by (3.1)-(3.2), and for some  $\delta_0 > 0$ ,

$$(3.3) \quad \operatorname{Im} \rho = -(1 + \mathcal{O}(e^{(\delta - \pi L)/\varepsilon})) \operatorname{Im} \int_0^{2\pi} \frac{\partial u_\varepsilon}{\partial r}(R, \theta) \bar{u}_\varepsilon(R, \theta) R d\theta$$

where the  $\mathcal{O}$  is locally uniform with respect to  $R$ .

Therefore, to prove our result, it is sufficient to obtain a lower bound on  $\operatorname{Im} \int_0^{2\pi} \frac{\partial u_\varepsilon}{\partial r}(R, \theta) \bar{u}_\varepsilon(R, \theta) R d\theta$ . Note that, by using (3.2), we immediately obtain (2.3).

#### 4. ESTIMATE OUTSIDE A LARGE DISC

The goal of this section is to prove,

**Proposition 4.1.** *Let  $R_1 > R_0 > 0$  be fixed in such a way that  $\bar{\mathcal{B}} \subset \{|(x, y)| < R_0\}$ . Then, for any  $C > 0$ , there exists a constant  $C' = C'(R_0, R_1, C) > 0$  such that, for all  $\varepsilon > 0$  small enough, one has,*

$$|\operatorname{Im} \rho| \geq \frac{1}{C'} \|u_\varepsilon\|_{L^2(R_0 < |(x, y)| < R_1)}^2 - C' e^{-C/\varepsilon}.$$

*Proof.* Working in polar coordinates  $(r, \theta)$ , for  $r \geq R_0$  we can represent  $u = u_\varepsilon$  as,

$$u(r, \theta) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} u_k(r) e^{ik\theta},$$

where  $u_k(r) := \int_0^{2\pi} u(r, \theta) e^{-ik\theta} d\theta = a_k H_k(r\sqrt{\rho})$ ,  $H_k$  being the outgoing Hankel function, defined as

$$H_k(t) := \frac{e^{i(t - \frac{k\pi}{2} - \frac{\pi}{4})}}{\Gamma(k + \frac{1}{2})} \sqrt{\frac{2}{\pi t}} \int_0^\infty e^{-s} s^{k - \frac{1}{2}} \left(1 + \frac{is}{2t}\right)^{k - \frac{1}{2}} ds,$$

and solution to,

$$t^2 H_k''(t) + t H_k'(t) + (t^2 - k^2) H_k(t) = 0.$$

In particular, for all  $k$ , the function  $h_k := H_k(r\sqrt{\rho})$  is an analytic function, solution to

$$(4.1) \quad -h_k'' - \frac{1}{r} h_k' + \frac{k^2}{r^2} h_k = \rho h_k,$$

and for any  $\mu > 0$  fixed small enough, one has,

$$(4.2) \quad h_k(re^{i\mu}) \in H^2([R_0, +\infty)).$$

By (3.3), for any  $R \in [R_0, R_1]$  we also have,

$$(4.3) \quad \operatorname{Im} \rho = -(1 + \mathcal{O}(e^{(\delta - \pi L)/\varepsilon})) \sum_{k \in \mathbb{Z}} \alpha_k(R) = -(1 + \mathcal{O}(e^{(\delta - \pi L)/\varepsilon})) \sum_{k \in \mathbb{Z}} \beta_k(R) |a_k|^2,$$

with

$$(4.4) \quad \alpha_k(R) := \operatorname{Im} R u_k'(R) \bar{u}_k(R) \quad ; \quad \beta_k(R) := \operatorname{Im} R h_k'(R) \bar{h}_k(R).$$

We set,

$$\lambda(R) := \sum_{k \in \mathbb{Z}} \alpha_k(R) = \sum_{k \in \mathbb{Z}} \beta_k(R) |a_k|^2,$$

and, for  $C > 0$  arbitrary large, we write,

$$\lambda(R) = \sum_{|k| \leq C/\varepsilon} \alpha_k(R) + \sum_{|k| > C/\varepsilon} \alpha_k(R) =: \lambda_-(R, C) + \lambda_+(R, C).$$

We first prove,

**Lemma 4.2.** *There exists  $\delta > 0$  such that, for any  $C > 0$ , one has,*

$$\lambda_+(R, C) = \mathcal{O}(e^{-\delta C/\varepsilon}),$$

uniformly as  $\varepsilon \rightarrow 0_+$ .

*Proof.* In view of (4.4), it is enough to prove that  $|u_k(R)| + |u'_k(R)| = \mathcal{O}(e^{-\delta|k|})$  for some  $\delta = \delta(R) > 0$ , uniformly as  $|k| \rightarrow \infty$ . From (4.1), we know that  $u_k$  is solution to,

$$-k^{-2}u_k'' - \frac{1}{k^2 r}u_k' + \frac{1}{r^2}u_k - \frac{\rho}{k^2}u_k = 0,$$

that can be considered as a semiclassical differential equation with small parameter  $h := |k|^{-1}$  and principal symbol  $a(r, r^*) := (r^*)^2 + r^{-2}$ . In particular, this symbol is locally elliptic, and since  $u$  is locally bounded together with all its derivatives, we also know that  $u_k$  is locally uniformly bounded (together with all its derivatives) as  $|k| \rightarrow \infty$ . Then, we can apply standard techniques of semiclassical analysis (in particular Agmon estimates: see, e.g., [Ma]) to prove that  $|u_k| + |u'_k|$  is locally  $\mathcal{O}(e^{-\delta|k|})$  for some  $\delta > 0$ , and the result follows.  $\square$

Next, we show,

**Lemma 4.3.** *For any  $C > 0$  and any  $\sigma \in (0, \pi L/2)$ , there exists  $C' = C'(C, \delta_1) > 0$  such that*

$$\lambda_-(R, C) \geq \frac{1}{C'} \sum_{|k| \leq C/\varepsilon} |a_k|^2 - C' |\operatorname{Im} \rho| e^{-2\sigma/\varepsilon},$$

uniformly as  $\varepsilon \rightarrow 0_+$ .

*Proof.* For  $|k| \leq C/\varepsilon$ , let  $\mu_k = \mu_{k,R} \in C^\infty(\mathbb{R}_+; \mathbb{R}_+)$  be a real non-decreasing function verifying,

$$\mu_k(r) = 0 \quad \text{for } r \leq r_k := \max(C_0|k|, R) \quad ; \quad \mu_k(r) = \frac{\mu_0}{1 + |k|} \quad \text{for } r \geq r_k + 1,$$

where  $\mu_0 > 0$  is fixed small enough, and  $C_0 > 0$  will be chosen sufficiently large later on. We set,

$$(4.5) \quad \nu_k(r) := r e^{i\mu_k(r)} \quad ; \quad g_k(r) = U_k h_k(r) := h_k(\nu_k(r)).$$

By (4.2) we have,

$$(4.6) \quad g_k \in H^2([R_0, +\infty)).$$

Moreover, by construction we also have,

$$\beta_k(R) = \operatorname{Im} \frac{\nu_k(R)}{\nu'_k(R)} g'_k(R) \bar{g}_k(R),$$

and, by using (4.1), we see that  $g_k$  is solution to,

$$(4.7) \quad -g_k'' - \left( \frac{\nu_k'}{\nu_k} - \frac{\nu_k''}{\nu_k'} \right) g_k' + \frac{k^2(\nu_k')^2}{\nu_k^2} g_k = \rho(\nu_k')^2 g_k.$$

Then, using (4.6)-(4.7), we can write,

$$\begin{aligned} \beta_k(R) &= -\operatorname{Im} \int_R^\infty \frac{d}{dr} \left( \frac{\nu_k(r)}{\nu_k'(r)} g_k'(r) \bar{g}_k(r) \right) dr \\ &= -\operatorname{Im} \int_R^\infty \left[ \left( 1 - \frac{\nu_k \nu_k''}{(\nu_k')^2} \right) g_k' \bar{g}_k + \frac{\nu_k(r)}{\nu_k'(r)} g_k'' \bar{g}_k(r) + \frac{\nu_k(r)}{\nu_k'(r)} |g_k'(r)|^2 \right] dr \\ &= -\operatorname{Im} \int_R^\infty \left[ \left( \frac{k^2 \nu_k'}{\nu_k} - \rho \nu_k \nu_k' \right) |g_k|^2 + \frac{\nu_k(r)}{\nu_k'(r)} |g_k'(r)|^2 \right] dr. \end{aligned}$$

Since  $\nu_k'/\nu_k = r^{-1} + i\mu_k'$  and  $\nu_k \nu_k' = r(1 + ir\mu_k')e^{2i\mu_k}$ , we obtain,

$$\beta_k(R) = \int_R^\infty (\gamma_k(r) |g_k'(r)|^2 + \delta_k(r) |g_k(r)|^2) dr,$$

with,

$$\begin{aligned} \gamma_k(r) &:= \frac{\mu_k'}{r^{-2} + (\mu_k')^2}; \\ \delta_k(r) &:= r \operatorname{Re} \rho \sin 2\mu_k + r \operatorname{Im} \rho \cos 2\mu_k \\ &\quad + r^2 \mu_k' [(\operatorname{Re} \rho) \cos 2\mu_k - (\operatorname{Im} \rho) \sin 2\mu_k] - k^2 \mu_k'. \end{aligned}$$

In particular,  $\gamma_k \geq 0$ . Since  $\mu_k \leq \mu_0(1 + |k|)^{-1}$ ,  $\operatorname{Im} \rho \leq 0$ , and  $\operatorname{Re} \rho \rightarrow \lambda_0 > 0$  as  $\varepsilon \rightarrow 0$ , we also have,

$$\delta_k \geq \delta_0 r \sin 2\mu_k + r \operatorname{Im} \rho \cos 2\mu_k + \mu_k'(\delta_0 r^2 - k^2),$$

where  $\delta_0$  is any positive constant such that  $\delta_0 < \lambda_0 \cos 2\mu_0$ . But, by construction, we have  $\mu_k'(r) = 0$  when  $r \leq C_0|k|$ . Therefore  $\mu_k'(r)(\delta_0 r^2 - k^2) \geq \mu_k'(r)(\delta_0 C_0^2 - 1)k^2 \geq 0$  if we choose  $C_0 \geq \delta_0^{-1/2}$ . Then, we obtain,

$$\begin{aligned} (4.8) \quad \beta_k(R) &\geq \int_R^\infty r (\delta_0 (\sin 2\mu_k(r) + \operatorname{Im} \rho \cos 2\mu_k(r))) |g_k(r)|^2 dr \\ &\geq \delta_0 \sin\left(\frac{\mu_0}{1 + |k|}\right) \int_{r_k+1}^\infty r |g_k(r)|^2 dr - |\operatorname{Im} \rho| \int_R^\infty r |g_k(r)|^2 dr. \end{aligned}$$

Since  $|k| \leq C/\varepsilon$  and  $|\operatorname{Im} \rho| = \mathcal{O}(e^{-c_1/\varepsilon})$  for some  $c_1 > 0$ , we also have  $|\operatorname{Im} \rho| \leq \frac{1}{2}\delta_0 \sin(\frac{\mu_0}{1 + |k|})$  for  $\varepsilon > 0$  small enough, and therefore,

$$\beta_k(R) \geq \frac{1}{2} \delta_0 \sin\left(\frac{\mu_0}{1 + |k|}\right) \int_{r_k+1}^\infty r |g_k(r)|^2 dr - |\operatorname{Im} \rho| \int_R^{r_k+1} r |g_k(r)|^2 dr.$$

Equivalently, setting  $v_k(r) := u_k(\nu_k(r)) = a_k g_k(r)$ , we have proved,

$$(4.9) \quad \alpha_k(R) \geq \frac{1}{2} \delta_0 |a_k|^2 \sin\left(\frac{\mu_0}{1 + |k|}\right) \int_{r_k+1}^\infty r |g_k(r)|^2 dr - |\operatorname{Im} \rho| \int_R^{r_k+1} r |v_k(r)|^2 dr$$

Now, considering a cut-off function  $\chi = \chi(r) \in C^\infty(\mathbb{R}_+; [0, 1])$  such that  $\chi = 1$  on  $r \geq R_0$ ,  $\chi = 0$  on  $r \leq R_0 - \delta_0$  ( $\delta_0 > 0$  small enough), we see that the function  $w := \chi u$  satisfies  $(-\Delta - \rho)w = [-\Delta, \chi]u$  on all of  $\mathbb{R}^2$ , and is outgoing. Then, standard estimates on the outgoing resolvent

of the Laplacian (or, equivalently, on the Green function of the Helmholtz equation in  $\mathbb{R}^n$ ,  $n \geq 2$ ) show that, for all  $\delta > 0$  arbitrarily small, one has  $w = \mathcal{O}(e^{\delta r} \|[-\Delta, \chi]u\|_{L^2})$  uniformly as  $r \rightarrow \infty$ . Actually, such estimates remain valid for the complex distorted Laplacian  $U_0 \Delta U_0^{-1}$  (where  $U_0$  is as in (4.5) with some arbitrary  $\mu_0 \geq 0$  small enough), and since  $\|[-\Delta, \chi]u\|_{L^2} = \mathcal{O}(e^{-\delta_1/\varepsilon})$  for any  $\delta_1 \in (0, \pi L/2)$ , we obtain:  $u(r) = \mathcal{O}(e^{\delta r - \delta_1/\varepsilon})$  uniformly on  $\{r \in \mathbb{C}; \operatorname{Re} r \geq R_0, |\operatorname{Im} r| \leq \mu_0(\operatorname{Re} R - R_0)\}$ , where  $\delta > 0$  is arbitrary. In particular, this gives us:  $r|v_k(r)|^2 = \mathcal{O}(e^{\delta r - 2\delta_1/\varepsilon})$ , and therefore,

$$\sum_{|k| \leq C/\varepsilon} \int_R^{r_k+1} r|v_k(r)|^2 dr = \mathcal{O}\left(\frac{C}{\varepsilon} e^{\delta C/\varepsilon - 2\delta_1/\varepsilon}\right) = \mathcal{O}(e^{-2\delta'_1/\varepsilon}),$$

where  $\delta'_1 = \delta_1 - \delta C$  can be taken arbitrarily close to  $\delta_1$  (and thus, to  $\pi L/2$ ) by choosing  $\delta \ll 1/C$ . Inserting into (4.9) and taking the sum over  $k$ , we obtain,

$$(4.10) \quad \lambda_-(R, C) \geq \frac{1}{2} \delta_0 \sum_{|k| \leq C/\varepsilon} |a_k|^2 \sin\left(\frac{\mu_0}{1+|k|}\right) \int_{r_k+1}^{\infty} r|g_k(r)|^2 dr - C' |\operatorname{Im} \rho| e^{-\delta'_1/\varepsilon}$$

with  $C' = C'(C) > 0$ .

In order to complete the proof, we need to estimate the quantity  $J_k := \int_{r_k+1}^{\infty} r|g_k(r)|^2 dr$  as  $|k| \rightarrow \infty$ . Setting  $r = |k|s$ , for  $|k|$  large enough we find,

$$(4.11) \quad J_k \geq |k|^2 \int_{2C_0}^{\infty} |w_k(s e^{i\mu_0/(1+|k|)})|^2 ds$$

where  $w_k(z) := z^{1/2} h_k(|k|z)$  ( $z \in \mathbb{C}$ ,  $|z| \geq C_0$ ,  $|\arg z| \leq \mu_0$ ). Using (4.1), we see that  $w_k$  is solution to,

$$-\frac{1}{k^2} w_k'' + \left( \frac{1}{z^2} - \frac{1}{4k^2 z^2} - \rho \right) w_k = 0.$$

This is a semiclassical Schrödinger equation, with small parameter  $h := |k|^{-1}$ , and we can apply to it the standard WKB complex method in order to find the asymptotic of  $w_k$ , both as  $k \rightarrow \infty$  and  $\operatorname{Re} z \rightarrow +\infty$ . Using also that  $w_k$  must be outgoing, we immediately obtain,

$$(4.12) \quad w_k(z) \sim \frac{\tau_k}{(\rho - z^{-2})^{1/4}} \exp\left(i|k| \int_{2C_0}^z (\rho - t^{-2})^{1/2} dt\right)$$

as  $|k| + \operatorname{Re} z \rightarrow \infty$ , uniformly with respect to  $\varepsilon > 0$ . Here  $\tau_k \in \mathbb{C}$  is a complex constant of normalization that we have to compute. In order to do so, we use the well-known asymptotic of  $H_k(t)$  as  $\operatorname{Re} t \rightarrow +\infty$ ,

$$H_k(t) \sim \sqrt{\frac{2}{\pi t}} \exp\left(i\left(t - \frac{k\pi}{2} - \frac{\pi}{4}\right)\right),$$

that gives,

$$w_k(r) = r^{1/2} H_k(|k|r\sqrt{\rho}) \sim \sqrt{\frac{2}{\pi|k|}} \exp\left(i\left(|k|r\sqrt{\rho} - \frac{k\pi}{2} - \frac{\pi}{4}\right)\right) \quad (r \rightarrow +\infty).$$



Comparing with (4.12), we obtain,

$$\tau_k = \rho^{\frac{1}{4}} \sqrt{\frac{2}{\pi|k|}} e^{-i(\frac{k\pi}{2} + \frac{\pi}{4})} e^{i|k|L}$$

where

$$L := \lim_{r \rightarrow +\infty} (r\sqrt{\rho} - \int_{2C_0}^r (\rho - t^{-2})^{\frac{1}{2}} dt) = \lim_{r \rightarrow +\infty} (r\sqrt{\rho} - [\sqrt{\rho t^2 - 1} - \tan^{-1} \sqrt{\rho t^2 - 1}]_{2C_0}^r)$$

that is,

$$L = \frac{\pi}{2} + \sqrt{4\rho C_0^2 - 1} - \tan^{-1} \sqrt{4\rho C_0^2 - 1}.$$

In particular,

$$\operatorname{Im} L = \operatorname{Im} \sqrt{4\rho C_0^2 - 1} + \frac{1}{2} \int_{\operatorname{Im} \sqrt{4\rho C_0^2 - 1}}^{-\operatorname{Im} \sqrt{4\rho C_0^2 - 1}} \frac{1}{1 + (\operatorname{Re} \sqrt{4\rho C_0^2 - 1} + it)^2} dt,$$

and thus

$$\operatorname{Im} L = (1 + \mathcal{O}(C_0^{-1})) \operatorname{Im} \sqrt{4\rho C_0^2 - 1} \leq 0$$

if  $C_0$  has been taken sufficiently large. As a consequence,

$$|\tau_k| \geq |\rho|^{\frac{1}{4}} \sqrt{\frac{2}{\pi|k|}},$$

and then, by (4.12), and for  $s \geq 2C_0$ , we deduce,

$$|k|^2 |w_k(s e^{i\mu_0/(1+|k|)})|^2 \geq \delta_2 |k| e^{-\delta s},$$

where  $\delta_2 > 0$  is a constant (independent both of  $k$  and  $\varepsilon$ ). Going back to (4.11), for  $|k|$  large enough we finally obtain,

$$J_k \geq \frac{|k|}{C_1},$$

where  $C_1$  is a positive constant. Then, inserting into (4.10), we obtain

$$\lambda_-(R, C) \geq \frac{\delta_0}{3C_1} \sum_{|k| \leq C/\varepsilon} |a_k|^2 - C' |\operatorname{Im} \rho| e^{-\delta'_1/\varepsilon},$$

and Lemma 4.3 follows.  $\square$

Now, for any  $K \geq 0$ , we have,

$$\|u\|_{r=R}^2 = R \sum_{k \in \mathbb{Z}} |a_k|^2 |h_k(R)|^2 \leq C_K \sum_{|k| \leq K} |a_k|^2 + R \sum_{|k| > K} |a_k|^2 |h_k(R)|^2,$$

with  $C_K := \sup_{|k| \leq K; R \in [R_0, R_1]} R |h_k(R)|^2$ . Then, we use the following elliptic estimate on the outgoing Hankel functions (see, e.g., [Bu1], Lemma 2.5): for any  $R' < R$ , there exists  $\delta = \delta(R, R') > 0$  such that,

$$|h_k(R)| = \mathcal{O}(e^{-\delta|k|} |h_k(R')|)$$

uniformly as  $|k| \rightarrow \infty$ . We obtain,

$$(4.13) \quad \|u\|_{r=R}^2 \leq C_K \sum_{|k| \leq K} |a_k|^2 + C e^{-2\delta K} \|u\|_{r=R'}^2,$$

where  $C = C(R, R')$  does not depend on  $K$ . Moreover, writing

$$\|u\|_{r=R'}^2 = \|u\|_{r=R}^2 - \int_{R'}^R 2 \operatorname{Re} \langle \partial_r u, u \rangle_{L^2(0, 2\pi)} dr,$$

for  $R_0 \leq R' < R \leq R_1$ , we obtain,

$$\|u\|_{r=R'}^2 = \|u\|_{r=R}^2 + \mathcal{O}(\|\partial_r u\|_{R_0 \leq r \leq R_1} \|u\|_{R_0 \leq r \leq R_1}),$$

and thus, using the equation  $-\Delta u = \rho u$  and standard Sobolev estimates,

$$\|u\|_{r=R'}^2 = \|u\|_{r=R}^2 + \mathcal{O}(\|u\|_{R_0 \leq r \leq R_1}^2).$$

Inserting this into (4.13), and taking  $K$  sufficiently large, we get

$$(4.14) \quad \|u\|_{r=R}^2 \leq C'_K \sum_{|k| \leq K} |a_k|^2 + C' e^{-2\delta K} \|u\|_{R_0 \leq r \leq R_1}^2,$$

where  $C', C'_K > 0$  are constants, and  $C'$  is independent of  $K$ . Finally, integrating from  $r = R_0$  to  $r = R_1$ , and increasing again the value of  $K$ , this gives,

$$(4.15) \quad \|u\|_{R_0 \leq r \leq R_1}^2 \leq 2C'_K \sum_{|k| \leq K} |a_k|^2.$$

Then, Proposition 4.1 directly follows from (4.3), Lemma 4.2, Lemma 4.3, and (4.15).  $\square$

**Remark 4.4.** By integrating with respect to  $R$  on any bounded interval of  $[R_0, +\infty)$ , and by using the equation  $-\Delta u_\varepsilon = \rho u_\varepsilon$  and standard estimates on the Laplacian, we easily deduce from this proposition that, for any bounded open set  $V \subset \{|(x, y)| \geq R_0\}$  and any  $s \geq 0$ , one has  $\|u_\varepsilon\|_{H^s(V)}^2 = \mathcal{O}(|\operatorname{Im} \rho| + e^{-C/\varepsilon})$  for any  $C > 0$ .

**Remark 4.5.** The result of Proposition 4.1 can easily be generalized to any dimension  $n \geq 2$  by working with the complex measure  $(\nu_k(r)/\nu'_k(r))^{n-1} dr$  instead of  $(\nu_k(r)/\nu'_k(r)) dr$  in the proof of Lemma 4.3.

**Remark 4.6.** As pointed out to us by J. Sjöstrand, an alternative (and probably more conceptual) proof of Proposition 4.1 may consists in making the change of scale  $r \mapsto r/h$ , where  $h > 0$  is an extra small parameter, and to apply the techniques of semiclassical analysis as  $h \rightarrow 0_+$ . The fact that  $u$  is outgoing means that it lives around the outgoing trajectories starting from the obstacle, and thus in a microlocal weighted space where  $-\hbar^2 \Delta - \rho$  can be written as the product of an elliptic pseudodifferential operator with  $\partial_r - iA$ , where the selfadjoint operator  $A$  acts on the tangent variable  $\theta$  only, and is positive. Such arguments are developed in [Sj], Section 4.

## 5. ESTIMATE NEAR THE OBSTACLE

Now, reasoning by contradiction, assume the existence of  $\delta_0 > 0$  such that, along a sequence  $\varepsilon \rightarrow 0^+$ , one has

$$(5.1) \quad |\operatorname{Im} \rho| = \mathcal{O}(e^{-(\pi L + \delta_0)/\varepsilon}).$$

In the rest of the proof, it will always be assumed that  $\varepsilon$  tends to zero along this sequence. Then Proposition 4.1 (added to standard Sobolev estimates) tells us that for any  $R_1 > R_0 > 0$  such that  $\bar{\mathcal{B}} \subset \{|(x, y)| < R_0\}$ , we have,

$$(5.2) \quad \|u_\varepsilon\|_{H^1(R_0 < |(x, y)| < R_1)}^2 = \mathcal{O}(e^{-(\pi L + \delta_0)/\varepsilon}).$$

To propagate this estimate up to an arbitrarily small neighborhood of  $\bar{\mathcal{B}}$ , we use the Carleman estimate in [LL, Theorem 3.5].

First fix a point  $(x_0, y_0)$  in  $\mathbf{E} = \mathbb{R}^2 \setminus \bar{\mathcal{B}}$ , and assume there exists a real function  $f$  defined on a small open neighborhood  $V_0$  of  $(x_0, y_0)$  in  $\mathbf{E}$ , with  $f(x_0, y_0) = 0$ ,  $\nabla f(x_0, y_0) \neq 0$ , and such that for any  $\delta > 0$  small enough, there exists  $\delta' = \delta'(\delta) > 0$ , such that,

$$(5.3) \quad \|u_\varepsilon\|_{H^1(V \cap \{f \geq \delta\})}^2 = \mathcal{O}(e^{-(\pi L + \delta')/\varepsilon}),$$

uniformly as  $\varepsilon \rightarrow 0_+$ . (For instance, in view of (5.2),  $(x_0, y_0)$  could be any point of  $\mathbf{E}$  such that  $|(x_0, y_0)| = R_-$ , with  $R_- := \inf\{R > 0; \bar{\mathcal{B}} \subset \{|(x, y)| \leq R\}\}$ , and  $f(x, y) = x^2 + y^2 - R_-^2$ .)

For  $\lambda > 0$  fixed large enough and  $(x, y)$  in  $V_0$ , following [LL, LR] we consider the function,

$$\varphi(x, y) := e^{\lambda(f(x, y) - (x - x_0)^2 - (y - y_0)^2)}.$$

Then, setting,

$$p_\varphi(x, y, \xi, \eta) := \xi^2 + \eta^2 - |\nabla \varphi(x, y)|^2 + 2i \langle \nabla \varphi(x, y), (\xi, \eta) \rangle = q_1 + iq_2,$$

it is easy to check that, if  $\lambda$  has been taken large enough, then there exists a constant  $C_0 > 0$  such that one has the implication,

$$p_\varphi(x, y, \xi, \eta) = 0 \Rightarrow \{q_1, q_2\}(x, y, \xi, \eta) \geq \frac{1}{C_0},$$

where  $\{q_1, q_2\}$  is the Poisson bracket of the real-valued functions  $q_1$  and  $q_2$ . Moreover, possibly by shrinking  $V_0$  around  $(x_0, y_0)$ , we see that  $\nabla \varphi \neq 0$  on  $V$ . In particular, Assumption 3.1 of [LL] is satisfied, and if  $\chi \in C_0^\infty(V_0; [0, 1])$  is such that  $\chi = 1$  near  $(x_0, y_0)$ , we can apply Theorem 3.5 of [LL] to the function  $w := \chi u_\varepsilon$ , and with small parameter  $h := \varepsilon/\mu$ , where  $\mu > 0$  is an extra-parameter that will be fixed small enough later on. Then, for  $\varepsilon/\mu$  small enough, we obtain,

$$(5.4) \quad \|e^{\mu\varphi/\varepsilon} w\|_{L^2}^2 + \mu^{-2} \varepsilon^2 \|e^{\mu\varphi/\varepsilon} \nabla w\|_{L^2}^2 \leq C \mu^{-3} \varepsilon^3 \|e^{\mu\varphi/\varepsilon} \Delta w\|_{L^2}^2$$

where  $C > 0$  is a constant. Then, writing  $-\Delta w = \rho w - [\Delta, \chi]u_\varepsilon$ , and observing that, for  $\varepsilon/\mu$  small enough, the term involving  $\rho w$  in the right-hand side of (5.4) can be absorbed by the first term of the left-hand side, we are led to,

$$\|e^{\mu\varphi/\varepsilon} w\|_{L^2}^2 + \mu^{-2} \varepsilon^2 \|e^{\mu\varphi/\varepsilon} \nabla w\|_{L^2}^2 \leq C \mu^{-3} \varepsilon^3 \|e^{\mu\varphi/\varepsilon} [\Delta, \chi]u_\varepsilon\|_{L^2}^2,$$

with a new constant  $C > 0$ . Now, setting  $m_0 := \sup_{V_0} \varphi$ ,  $V'_0 := \{\chi = 1\}$ ,  $S_\delta := \text{Supp} \nabla \chi \cap \{f < \delta\}$  ( $\delta > 0$  small enough), and using (5.3), we deduce,

$$(5.5) \quad \begin{aligned} & \|e^{\mu\varphi/\varepsilon} u_\varepsilon\|_{L^2(V'_0)}^2 + \mu^{-2} \varepsilon^2 \|e^{\mu\varphi/\varepsilon} \nabla u_\varepsilon\|_{L^2(V'_0)}^2 \\ &= \mathcal{O}(\mu^{-3} \varepsilon^3 \|e^{\mu\varphi/\varepsilon} [\Delta, \chi]u_\varepsilon\|_{L^2(S_\delta)}^2 + e^{(\mu m_0 - \pi L - \delta')/\varepsilon}). \end{aligned}$$

On the other hand, we have  $S_\delta \subset \{f < \delta\} \cap \{|(x, y) - (x_0, y_0)| \geq \delta_1\}$  for some  $\delta_1 > 0$  independent of  $\delta$ , and thus, by construction, for  $\delta > 0$  sufficiently small, there exists a constant  $\delta_2 > 0$  such that,

$$(5.6) \quad S_\delta \subset \{\varphi(x, y) \leq 1 - \delta_2\}.$$

As a consequence, we obtain,

$$(5.7) \quad \begin{aligned} & \|e^{\mu\varphi/\varepsilon} u_\varepsilon\|_{L^2(V'_0)}^2 + \mu^{-2} \varepsilon^2 \|e^{\mu\varphi/\varepsilon} \nabla u_\varepsilon\|_{L^2(V'_0)}^2 \\ &= \mathcal{O}(\mu^{-3} \varepsilon^3 e^{\mu(1-\delta_2)/\varepsilon} \|u_\varepsilon\|_{H^1(S_\delta)}^2 + e^{(\mu m_0 - \pi L - \delta')/\varepsilon}). \end{aligned}$$

Since  $S_\delta \subset \mathbf{E}$ , we also know (see (3.2)) that  $\|u_\varepsilon\|_{H^1(S)}$  is not exponentially larger than  $e^{-\pi L/2\varepsilon}$ . Moreover, since  $\varphi(x_0, y_0) = 1$ , if  $B_r$  stands for the ball of radius  $r$  centered at  $(x_0, y_0)$ , we have  $\varphi \leq 1 - \theta(r)$  on  $B_r$ , with  $\theta(r) \rightarrow 0$  as  $r \rightarrow 0$ . Therefore, for  $r > 0$  small enough, we deduce from (5.7),

$$(5.8) \quad \begin{aligned} & \|u_\varepsilon\|_{L^2(B_r)}^2 + \mu^{-2} \varepsilon^2 \|\nabla u_\varepsilon\|_{L^2(B_r)}^2 \\ &= \mathcal{O}(\mu^{-3} \varepsilon^3 e^{(\mu(\theta(r) - \frac{1}{2}\delta_2) - \pi L)/\varepsilon} + e^{(\mu(m_0 - 1 + \theta(r)) - \pi L - \delta')/\varepsilon}). \end{aligned}$$

Now, we first fix  $\delta > 0$  such that (5.6) is satisfied, and then  $r > 0$  and  $\mu > 0$  sufficiently small, in such a way that  $\theta(r) \leq \frac{1}{4}\delta_2$  and  $(\mu(m_0 - 1 + \theta(r))) \leq \frac{1}{2}\delta'$ . We obtain,

$$\|u_\varepsilon\|_{L^2(B_r)}^2 + \varepsilon^2 \|\nabla u_\varepsilon\|_{L^2(B_r)}^2 = \mathcal{O}(e^{-\pi L/\varepsilon} (e^{-\frac{\mu}{4}\delta_2/\varepsilon} + e^{-\frac{1}{2}\delta'/\varepsilon}))$$

In other words, we have extended the estimate (5.3) across the boundary  $\{f = 0\}$  near  $(x_0, y_0)$ . Our argument can be performed near any point  $(x_0, y_0) \in \mathbf{E}$  where an estimate like (5.3) is valid, and thus, starting from the points of the circle  $\{|(x, y)| = R_-\}$  (where the estimate is valid thanks to Proposition 4.1 and to the assumption (5.1)), and deforming continuously this circle up to make it become the boundary of  $\mathcal{B}$ , a standard covering argument leads to,

**Proposition 5.1.** *Under assumption (5.1), for any compact set  $K \subset \mathbf{E}$ , there exists  $\delta = \delta(K) > 0$  such that,*

$$\|u_\varepsilon\|_{H^1(K)}^2 = \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon}),$$

uniformly as  $\varepsilon \rightarrow 0_+$ .

**Remark 5.2.** *By using the equation, we deduce that, actually, in the previous estimate  $H^1$  can be replaced by any  $H^m$ ,  $m \geq 0$ .*

## 6. ESTIMATE AT THE BOUNDARY

Now, we plan to propagate the estimates of the previous section up to the boundary of  $\mathcal{B}$  (but away from any arbitrarily small neighborhood of  $M_0$ ), by making use of the Carleman estimate at the boundary as stated in [LR], Proposition 2 (see also [LL], Theorem 7.6, applied to  $e^{-\rho t} u_\varepsilon(x, y)$ ).

We consider an arbitrary point  $(x_0, y_0)$  on the boundary  $\partial\mathcal{B}$  of  $\mathcal{B}$ , with  $(x_0, y_0) \neq (L, 0)$ , and a small enough open neighborhood  $V$  of  $(x_0, y_0)$  in  $\mathbb{R}^2$ .

We also consider a compact neighborhood  $K \subset V$  of  $(x_0, y_0)$ , and we denote by  $f$  a function defining  $\partial\mathcal{B}$  near  $(x_0, y_0)$ , in the sense that one has,

$$\mathcal{B} \cap V = \{(x, y) \in V; f(x, y) < 0\},$$

and  $\nabla f \neq 0$  on  $V$ . Finally, as in following [LL, LR], one sets,

$$\varphi(x, y) := e^{\lambda(f(x, y) - (x - x_0)^2 - (y - y_0)^2)},$$

where  $\lambda > 0$  is fixed sufficiently large and  $C_0 > \sup_V (f(x, y) - (x - x_0)^2 - (y - y_0)^2)$ . In particular, if  $V$  has been taken sufficiently small, we see (e.g. as in [LL], Lemma A.1) that  $\varphi$  satisfies Assumption (8) of [LR]. Moreover, since the outward pointing unit normal to  $\mathbf{E}$  in  $V$  is  $n := -\nabla f / |\nabla f|$ , we also have  $\partial_n \varphi|_{\partial\mathbf{E} \cap V} < 0$ . Therefore, we can apply Proposition 2 of [LR] (or, alternatively, Theorem 7.6 of [LL]), and we obtain the existence of a constant  $C > 0$  such that, for any  $\mu, \varepsilon > 0$  with  $\varepsilon/\mu$  small enough,

$$\begin{aligned} \|e^{\mu\varphi/\varepsilon} \chi u_\varepsilon\|_{L^2(\mathbf{E} \cap V)}^2 + \mu^{-2} \varepsilon^2 \|e^{\mu\varphi/\varepsilon} \nabla(\chi u_\varepsilon)\|_{L^2(\mathbf{E} \cap V)}^2 \\ \leq C \mu^{-3} \varepsilon^3 \|e^{\mu\varphi/\varepsilon} \Delta(\chi u_\varepsilon)\|_{L^2(\mathbf{E} \cap V)}^2, \end{aligned}$$

where  $\chi \in C_0^\infty(V; [0, 1])$  is some fixed cut-off function such that  $\chi = 1$  on  $K$ . Using that  $-\Delta u_\varepsilon = \rho u_\varepsilon$ , for  $\varepsilon$  small enough, we deduce,

$$\|e^{\mu\varphi/\varepsilon} u_\varepsilon\|_{L^2(\mathbf{E} \cap K)}^2 + \mu^{-2} \varepsilon^2 \|e^{\mu\varphi/\varepsilon} \nabla u_\varepsilon\|_{L^2(\mathbf{E} \cap K)}^2 \leq 2C \mu^{-3} \varepsilon^3 \|e^{\mu\varphi/\varepsilon} [\Delta, \chi] u_\varepsilon\|_{L^2(\mathbf{E} \cap V)}^2.$$

Now, for all  $\delta > 0$  small enough, on  $\text{Supp} \nabla \chi \cap \{f \leq \delta\} \cap V$ , we have,

$$\varphi \leq \varphi(x_0, y_0) - \delta',$$

with  $\delta' = \delta'(\delta) > 0$ . On the other hand, on  $\{f \geq \delta\} \cap V$ , by Proposition 5.1 we have,

$$\|u_\varepsilon\|_{L^2(\{f \geq \delta\} \cap V)}^2 = \mathcal{O}(e^{-(\pi L + \delta')/\varepsilon}).$$

Therefore, using also (3.2), and fixing  $\mu > 0$  in a convenient way as before, we obtain the existence of  $\delta_1 > 0$ , such that,

$$\|e^{\mu\varphi/\varepsilon} u_\varepsilon\|_{L^2(\mathbf{E} \cap K)}^2 + \varepsilon^2 \|e^{\mu\varphi/\varepsilon} \nabla u_\varepsilon\|_{L^2(\mathbf{E} \cap K)}^2 = \mathcal{O}(e^{(\mu\varphi(x_0, y_0) - \pi L - \delta_1)/\varepsilon}),$$

and, if  $V' \subset K$  is a sufficiently small neighborhood of  $(x_0, y_0)$ , we finally obtain,

$$\|u_\varepsilon\|_{H^1(\mathbf{E} \cap V')}^2 = \mathcal{O}(e^{-(\pi L + \frac{1}{2}\delta_1)/\varepsilon}).$$

Since  $(x_0, y_0)$  was arbitrary on  $\partial\mathcal{B} \setminus \{M_0\}$  (where  $M_0 = (L, 0)$ ), we have proved,

**Proposition 6.1.** *Under the assumption (5.1), for any neighborhood  $\mathcal{U}$  of  $M_0$  and any compact set  $K \subset \mathbb{R}^2$ , there exists  $\delta > 0$  such that,*

$$\|u_\varepsilon\|_{H^1(\mathbf{E} \cap K \setminus \mathcal{U})}^2 = \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon}),$$

uniformly as  $\varepsilon \rightarrow 0_+$ .

**Remark 6.2.** *By using the equation and a standard result of regularity on the Dirichlet Laplacian (see, e.g., [Br]), we can deduce that, in the previous estimate,  $H^1$  can be replaced by any  $H^m$ ,  $m \geq 0$ .*

## 7. ESTIMATE NEAR THE APERTURE

Now, we concentrate our attention to a small neighborhood of  $M_0$  in  $\overline{\mathbf{E}}$ . More precisely, we fix  $\varepsilon_1 \in (0, \varepsilon_0]$ , such that,

$$\frac{\pi^2}{4\varepsilon_1^2} > \lambda_0,$$

and we consider the rectangle,

$$Q := [L, L + \varepsilon_1] \times [-\varepsilon_1, \varepsilon_1].$$

In particular,  $M_0$  belongs to  $\partial Q$ , and, if  $\varepsilon_1$  is taken sufficiently small, then,

$$Q \setminus (\{L\} \times [-\varepsilon_1, \varepsilon_1]) \subset \mathbf{E}.$$

Moreover, by Proposition 6.1, we know the existence of some  $\delta > 0$  such that  $u_\varepsilon$  is  $\mathcal{O}(e^{-(\pi L + \delta)/\varepsilon})$  near  $\partial Q \setminus (\{L\} \times [-\varepsilon_1, \varepsilon_1])$ , and, by (2.1), we also have  $u_\varepsilon = 0$  on  $(\{L\} \times [-\varepsilon_1, -\varepsilon]) \cup (\{L\} \times [\varepsilon, \varepsilon_1])$ .

Let  $\chi \in C_0^\infty(\mathbb{R}^2; [0, 1])$  such that (see Figure 2),

- $\chi = 1$  on  $[L, L + \frac{1}{2}\varepsilon_1] \times [-\frac{1}{2}\varepsilon_1, \frac{1}{2}\varepsilon_1]$ ;
- $\chi = 0$  on  $([L + \varepsilon_1, +\infty) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, -\varepsilon_1]) \cup (\mathbb{R} \times [\varepsilon_1, +\infty))$ .

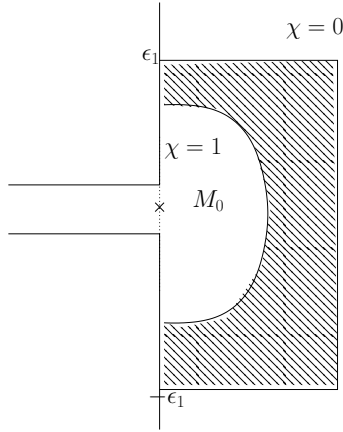


FIGURE 2. The aperture.

We set,

$$v := \chi u_\varepsilon.$$

In particular,  $v \in H^2(Q)$ , and  $v|_{|y|=\varepsilon_1} = 0$ . Therefore, on  $Q$ , we can expand  $v$  as,

$$(7.1) \quad v(x, y) = \sum_{j \geq 1} v_j(x) \varphi_j(y),$$

where the  $\varphi_j$ 's are the eigenfunctions of the Dirichlet realization of  $-d^2/dy^2$  on  $[-\varepsilon_1, \varepsilon_1]$ , namely,

$$\varphi_{2j}(y) = \frac{1}{\sqrt{\varepsilon_1}} \sin(\alpha_{2j}y/\varepsilon_1); \quad \varphi_{2j-1}(y) = \frac{1}{\sqrt{\varepsilon_1}} \cos(\alpha_{2j-1}y/\varepsilon_1); \quad \alpha_j := \frac{j\pi}{2},$$

and  $v_j \in H^2([L, L + \varepsilon_1])$ . Moreover, using Proposition 6.1 and Remark 6.2, on  $Q$  we have,

$$-\Delta v = \rho v + r$$

where  $\|r\|_{H^m(Q)}^2 = \|[\Delta, \chi]u_\varepsilon\|_{H^m(Q)}^2 = \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon})$ , and  $r|_{|y|=\varepsilon_1} = 0$  ( $m \geq 0$  arbitrary, and  $\delta = \delta(m) > 0$ ). We deduce that the  $v_j$ 's verify,

$$(7.2) \quad -v_j'' + \beta_j v_j = r_j,$$

where we have set  $\beta_j := \frac{\alpha_j^2}{\varepsilon_1^2} - \rho$ , and  $r_j := \int_{-\varepsilon_1}^{\varepsilon_1} r(x, y) \varphi_j(y) dy$ , so that we have,

$$(7.3) \quad \sum_{j \geq 1} j^m \|r_j\|_{H^m([L, L + \varepsilon_1])}^2 = \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon}).$$

By construction, we also have  $v_j = 0$  on  $[L + \varepsilon_1, +\infty)$ .

**Proposition 7.1.** *Assume (5.1). Then, for all  $j \geq 1$ , there exist  $b_j \in \mathbb{C}$  and  $s_j \in \cap_{m \geq 0} H^m([L, L + \varepsilon_1])$ , such that,*

$$v_j(x) = b_j e^{-(x-L)\sqrt{\beta_j}} + s_j(x);$$

$$\sum_{j \geq 1} j^m \|s_j\|_{H^m([L, L + \varepsilon_1])}^2 = \mathcal{O}(e^{-(\pi L + \delta_m)/\varepsilon}),$$

with  $\delta_m > 0$  and uniformly with respect to  $\varepsilon$  small enough.

*Proof.* Set,

$$W_j := \begin{pmatrix} v_j \\ v_j' \end{pmatrix}.$$

Then, by (7.2),  $W_j$  is solution of,

$$\begin{cases} W_j' = A_j W_j - R_j; \\ W_j(L + \varepsilon_1) = 0, \end{cases}$$

with  $A_j := \begin{pmatrix} 0 & 1 \\ \beta_j & 0 \end{pmatrix}$  and  $R_j := \begin{pmatrix} 0 \\ r_j \end{pmatrix}$ . Therefore,

$$W_j(x) = \int_x^{L+\varepsilon_1} e^{(x-t)A_j} R_j(t) dt,$$

and, diagonalizing  $A_j$  and re-writing the solution in a basis of eigenvectors of  $A_j$ , we obtain in particular,

$$v_j'(x) + \sqrt{\beta_j} v_j(x) = \int_x^{L+\varepsilon_1} e^{(x-t)\sqrt{\beta_j}} r_j(t) dt.$$

Using again that  $v(L + \varepsilon_1) = 0$ , we deduce,

$$v_j(x) = - \int_x^{L+\varepsilon_1} \int_{x_1}^{L+\varepsilon_1} e^{(2x_1-t-x)\sqrt{\beta_j}} r_j(t) dt dx_1.$$

Then, the results follows with  $b_j := - \int_L^{L+\varepsilon_1} \int_{x_1}^{L+\varepsilon_1} e^{(2x_1-t-L)\sqrt{\beta_j}} r_j(t) dt dx_1$  and  $s_j(x) := \int_L^x \int_{x_1}^{L+\varepsilon_1} e^{(2x_1-t-x)\sqrt{\beta_j}} r_j(t) dt dx_1$ , by observing that  $\operatorname{Re}((2x_1 - t - x)\sqrt{\beta_j}) < 0$  on the domain of integration of  $s_j(x)$  and by using (7.3).  $\square$

**Remark 7.2.** Let  $\varepsilon_2 \in (0, \frac{1}{2}\varepsilon_1)$  arbitrary. By Proposition 5.1, we know that there exists a constant  $\delta = \delta(\varepsilon_2) > 0$  such that,

$$\|v\|_{L^2((L+\varepsilon_2, L+\varepsilon_1) \times (-\varepsilon_1, \varepsilon_1))} = \mathcal{O}(e^{-(\pi L + \delta)/2\varepsilon}).$$

On the other hand, using (7.1) and Proposition 7.1, on  $(L, L+\varepsilon_1) \times (-\varepsilon_1, \varepsilon_1)$ , we have,

$$v(x, y) = \sum_{j \geq 1} b_j e^{-(x-L)\sqrt{\beta_j}} \varphi_j(y) + s(x, y),$$

with  $\|s\|_{L^2((L, L+\varepsilon_1) \times (-\varepsilon_1, \varepsilon_1))} = \mathcal{O}(e^{-(\pi L + \delta_0)/2\varepsilon})$  for some constant  $\delta_0 > 0$ . Since  $\sqrt{\beta_j} \sim \frac{j\pi}{2\varepsilon_1}$  as  $j \rightarrow \infty$ , and  $\varepsilon_2$  is arbitrarily small, we immediately deduce that, for any  $\nu > 0$ , there exists  $\delta = \delta(\nu) > 0$ , such that,

$$(7.4) \quad \sum_{j \geq 1} |b_j|^2 e^{-\nu j} = \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon}),$$

uniformly as  $\varepsilon \rightarrow 0_+$ .

## 8. REPRESENTATIONS AT THE APERTURE

In this section, we consider the trace of  $v$  on  $\{x = L\}$ . By construction, it also coincides with the trace  $u_\varepsilon$  as long as  $|y| < \frac{1}{2}\varepsilon_1$ . Now, as in [MN], there are two ways of taking this trace, depending if one takes the limit  $x \rightarrow L_+$  or  $x \rightarrow L_-$ .

Considering first the limit  $x \rightarrow L_-$ , we can just apply the results of [MN], Sections 4 & 6 (in particular (4.2), (4.3) and Lemma 6.1), and, for  $x < L$  close to  $L$  and  $|y| < \varepsilon$ , we obtain,

$$(8.1) \quad v(x, y) = \sum_{k=1}^{\infty} \left( a_{k,+} e^{\theta_k x / \varepsilon} + a_{k,-} e^{-\theta_k x / \varepsilon} \right) \psi_k(y),$$

where we have used the notations,

$$\begin{aligned} \psi_{2k}(y) &= \frac{1}{\sqrt{\varepsilon}} \sin(\alpha_{2k} y / \varepsilon); \quad \psi_{2k-1}(y) = \frac{1}{\sqrt{\varepsilon}} \cos(\alpha_{2k-1} y / \varepsilon); \quad \alpha_k := \frac{k\pi}{2}; \\ \theta_k &:= \sqrt{\alpha_k^2 - \varepsilon^2 \rho(\varepsilon)}, \end{aligned}$$

(here  $\sqrt{\cdot}$  stands for the principal square root), and where  $a_{k,\pm}$  are ( $\varepsilon$ -dependent) constant complex numbers. Moreover, the sum converges in



$H^2((L - \varepsilon_1, L) \times (-\varepsilon, \varepsilon))$ , and the limit  $x \rightarrow L_-$  gives (see [MN], Lemma 6.1),

$$(8.2) \quad v(L, y) = \sum_{k=1}^{\infty} \left( a_{k,+} e^{\theta_k L/\varepsilon} + a_{k,-} e^{-\theta_k L/\varepsilon} \right) \psi_k(y),$$

together with (see [MN], formula (6.7)),

$$(8.3) \quad \partial_x v(L, y) = \frac{1}{\varepsilon} \sum_{k \geq 1} \theta_k \left( a_{k,+} e^{\theta_k L/\varepsilon} - a_{k,-} e^{-\theta_k L/\varepsilon} \right) \psi_k(y) \text{ in } H^{1/2}(|y| \leq \varepsilon).$$

Then, starting from (7.1), and using similar arguments, the limit  $x \rightarrow L_+$  can be taken in the same way, and, using Proposition 7.1, we obtain,

$$(8.4) \quad v(L, y) = \sum_{j=1}^{\infty} (b_j + s_j(L)) \varphi_j(y),$$

together with,

$$(8.5) \quad \partial_x v(L, y) = \sum_{j=1}^{\infty} (-\sqrt{\beta_j} b_j + s_j'(L)) \varphi_j(y) \text{ in } H^{1/2}(|y| \leq \varepsilon_1).$$

Moreover, still by Proposition 7.1, we have,

$$(8.6) \quad \sum_{j \geq 1} (|s_j(L)|^2 + |s_j'(L)|^2) = \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon}),$$

for some constant  $\delta > 0$ .

## 9. ESTIMATES ON THE COEFFICIENTS

At this point, we can proceed as [MN], Section 7 (but working with  $v$  instead of  $u_\varepsilon$ ), with the difference that, in our present case, the index  $j_0$  appearing in [MN], formula (6.8), is just 0 (that is, all the sums over  $\{j \leq j_0\}$  become null). For the sake of completeness, we briefly reproduce these arguments here.

The main idea consists in computing in two different ways the three following quantities:

$$\langle v, \partial_x v \rangle_{\{L\} \times [-\varepsilon_1, \varepsilon_1]}, \quad \langle v, \varphi_1 \rangle_{\{L\} \times [-\varepsilon_1, \varepsilon_1]}, \quad \langle \partial_x v, \psi_1 \rangle_{\{L\} \times [-\varepsilon, \varepsilon]}.$$

We set

$$A_{k,\pm} := a_{k,\pm} e^{\pm \theta_k L/\varepsilon}.$$

In view of (8.2)-(8.6), and since  $v(L, y)$  vanishes identically on  $\{\varepsilon < |y| < \varepsilon_1\}$ , the two computations of  $\langle v, \partial_x v \rangle_{\{L\} \times [-\varepsilon_1, \varepsilon_1]}$  give the identity

$$\frac{1}{\varepsilon} \sum_{k \geq 1} \theta_k (|A_{k,+}|^2 - |A_{k,-}|^2 + 2i \operatorname{Im}(A_{k,+} \bar{A}_{k,-})) = - \sum_{j \geq 1} (\sqrt{\beta_j} |b_j|^2 + r(\varepsilon)),$$

with

$$\begin{aligned}
 r(\varepsilon) &= \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon} + e^{-(\pi L + \delta)/2\varepsilon} (\sum_{j \geq 1} |b_j|^2)^{\frac{1}{2}}) \\
 (9.1) \quad &= \mathcal{O}(e^{-(\pi L + \frac{\delta}{2})/\varepsilon} + e^{-\delta/\varepsilon} \sum_{j \geq 1} |b_j|^2).
 \end{aligned}$$

Taking the real part, and using the fact that  $\operatorname{Re} \theta_k \sim k\pi/2$  as  $k \rightarrow \infty$ , while  $|\operatorname{Im} \theta_k| = \mathcal{O}(k^{-1}e^{-\delta/\varepsilon})$  for some constant  $\delta > 0$ , we obtain,

$$\begin{aligned}
 \frac{1}{\varepsilon} \sum_{k \geq 1} (\operatorname{Re} \theta_k) (|A_{k,+}|^2 - |A_{k,-}|^2) + \frac{1}{\varepsilon} \sum_{k \geq 1} \mathcal{O}(k^{-1}e^{-\delta/\varepsilon}) |A_{k,+} A_{k,-}| \\
 = - \sum_{j \geq 1} (\operatorname{Re} \sqrt{\beta_j}) |b_j|^2 + r(\varepsilon).
 \end{aligned}$$

In particular, since  $\operatorname{Re} \sqrt{\beta_j} = \frac{\pi j}{2\varepsilon_1} (1 + \mathcal{O}(\varepsilon^2 j^{-2}))$ , we see that there exists a constant  $C > 0$  such that

$$\begin{aligned}
 \sum_{k \geq 1} \operatorname{Re} \theta_k (|A_{k,+}|^2 - |A_{k,-}|^2) &\leq C \sum_{k \geq 1} k^{-1} e^{-\delta/\varepsilon} |A_{k,+} A_{k,-}| \\
 (9.2) \quad &- \frac{\pi}{2} \frac{\varepsilon}{\varepsilon_1} \sum_{j \geq 1} j (1 - C\varepsilon^2 j^{-2}) |b_j|^2 + r(\varepsilon).
 \end{aligned}$$

Moreover, by Appendix A in [MN], there exists a constant  $c > 0$ , such that,

$$(9.3) \quad \sum_{k \geq 1} k |a_{k,-} e^{-c\theta_k}|^2 = \mathcal{O}(\varepsilon^{-1/2}),$$

and thus, for  $\varepsilon$  small enough,

$$(9.4) \quad \sum_{k \geq 2} k |A_{k,-}|^2 = \sum_{k \geq 2} k |a_{k,-} e^{-c\theta_k}|^2 e^{-2\theta_k(\frac{L}{\varepsilon} - c)} = \mathcal{O}(\varepsilon^{-1/2} e^{-2\pi L/\varepsilon}).$$

Therefore, we deduce from (9.2) (with some new positive constants  $C, \delta$ ),

$$\begin{aligned}
 \sum_{k \geq 1} (k - Ck^{-1}e^{-\delta/\varepsilon}) |A_{k,+}|^2 \\
 (9.5) \quad \leq (1 + Ce^{-\delta/\varepsilon}) |A_{1,-}|^2 - \frac{2\varepsilon}{\pi} (1 + r_1(\varepsilon)) \sum_{j \geq 1} \operatorname{Re} \sqrt{\beta_j} |b_j|^2 + r_2(\varepsilon),
 \end{aligned}$$

with

$$(9.6) \quad r_1(\varepsilon) = \mathcal{O}(e^{-\delta/\varepsilon}) \quad ; \quad r_2(\varepsilon) = \mathcal{O}(e^{-(\pi L + \delta)/\varepsilon}).$$

Now, computing the scalar products  $\langle v(L, \cdot), \varphi_1 \rangle$  and  $\langle \partial_x v(L, \cdot), \psi_1 \rangle_{L^2(|y| < \varepsilon)}$  in two different ways (by using (8.2)-(8.6) and the fact that  $v(L, y) = 0$  on  $\{\varepsilon < |y| < \varepsilon_1\}$ ), we find

$$\begin{aligned}
 \sum_{k \geq 1} \mu_k (A_{k,+} + A_{k,-}) &= b_1; \\
 \frac{1}{\varepsilon} \theta_1 (A_{1,+} - A_{1,-}) &= - \sum_{j \geq 1} \nu_j (\sqrt{\beta_j} b_j - s'_j(L)),
 \end{aligned}$$

with

$$\mu_k := \int_{-\varepsilon}^{\varepsilon} \psi_k(y) \varphi_1(y) dy = \begin{cases} 0 & \text{if } k \text{ is even;} \\ (-1)^{\frac{k-1}{2}} \frac{4k\sqrt{\varepsilon/\varepsilon_1}}{\pi(k^2 - (\varepsilon/\varepsilon_1)^2)} \cos \frac{\pi}{2} \frac{\varepsilon}{\varepsilon_1} & \text{if } k \text{ is odd,} \end{cases}$$

and

$$\nu_j := \int_{-\varepsilon}^{\varepsilon} \varphi_j(y) \psi_1(y) dy = \begin{cases} 0 & \text{if } j \text{ is even;} \\ \frac{4\sqrt{\varepsilon/\varepsilon_1} \sin(((\varepsilon/\varepsilon_1)j-1)\pi/2)}{\pi((\varepsilon/\varepsilon_1)^2 j^2 - 1)} & \text{if } j \neq \frac{\varepsilon_1}{\varepsilon} \text{ is odd;} \\ \sqrt{\varepsilon/\varepsilon_1} & \text{if } j = \frac{\varepsilon_1}{\varepsilon} \text{ is odd.} \end{cases}$$

Using (9.4) again and (7.4), we obtain

$$(9.7) \quad |A_{1,+} + A_{1,-}| \leq C e^{-(\pi L + \delta)/2\varepsilon} + \sum_{k \geq 2} \left| \frac{\mu_k}{\mu_1} A_{k,+} \right| + \frac{C}{\sqrt{\varepsilon}} e^{-\pi L/\varepsilon};$$

$$(9.8) \quad |A_{1,+} - A_{1,-}| \leq \frac{\varepsilon}{|\theta_1|} \sum_{j \geq 1} |\nu_j \sqrt{\beta_j} b_j| + C e^{-(\pi L + \delta)/2\varepsilon},$$

with some new constant  $C > 0$ .

Then, we observe that  $|\mu_k/\mu_1| \leq (k - \frac{\varepsilon^2}{\varepsilon_1^2})^{-1}$  ( $k$  odd), thus by (9.5),

$$(9.9) \quad \begin{aligned} \sum_{k \geq 2} \left| \frac{\mu_k}{\mu_1} A_{k,+} \right| &\leq \left( \sum_{k \geq 3} \frac{1}{k(k - \frac{\varepsilon^2}{\varepsilon_1^2})^2} \right)^{\frac{1}{2}} \left( \sum_{k \geq 2} k |A_{k,+}|^2 \right)^{\frac{1}{2}} \\ &\leq \tau_1 \left( \alpha |A_{1,-}|^2 - \beta \frac{2\varepsilon}{\pi} \sum_{j \geq 1} \operatorname{Re} \sqrt{\beta_j} |b_j|^2 + r_2(\varepsilon) \right)^{\frac{1}{2}} + C e^{-(\pi L + \delta)/2\varepsilon}, \end{aligned}$$

where  $\tau_1$  can be taken arbitrarily close to  $(\sum_{k \geq 3} k^{-3})^{\frac{1}{2}} < \frac{1}{2}$ , and  $\alpha, \beta$  are positive numbers that tend to 1 as  $\varepsilon \rightarrow 0$ , and are such that  $\alpha |A_{1,-}|^2 - \beta \frac{2\varepsilon}{\pi} \sum_{j \geq 1} \operatorname{Re} \sqrt{\beta_j} |b_j|^2 + r_2(\varepsilon)$  remains non negative for all  $\varepsilon > 0$  small enough. Inserting (9.9) into (9.7), we obtain

(9.10)

$$|A_{1,+} + A_{1,-}| \leq \tau_1 \left( \alpha |A_{1,-}|^2 - \beta \frac{2\varepsilon}{\pi} \sum_{j \geq 1} \operatorname{Re} \sqrt{\beta_j} |b_j|^2 + r_2(\varepsilon) \right)^{\frac{1}{2}} + 2C e^{-(\pi L + \delta)/2\varepsilon}.$$

On the other hand, going back to (9.8), the Cauchy-Schwarz inequality gives,

$$(9.11) \quad \frac{\varepsilon}{|\theta_1|} \sum_{j \geq 1} |\nu_j \sqrt{\beta_j} b_j| \leq \tau_2 \left( \frac{2\varepsilon}{\pi} \sum_{j \geq 1} |b_j|^2 |\sqrt{\beta_j}| \right)^{\frac{1}{2}}$$

with

$$\begin{aligned}
 \tau_2^2 &= \frac{\varepsilon\pi}{2|\theta_1|^2} \sum_{j \geq 1} j |\nu_j|^2 |\sqrt{\beta_j}| \\
 (9.12) \quad &= \frac{16}{\pi^2} (1 + \mathcal{O}(\varepsilon^2)) \sum_{j \geq 1, j \text{ odd}} \frac{\varepsilon}{\varepsilon_1} \frac{\sin^2 \left( \left( \frac{j\varepsilon}{\varepsilon_1} - 1 \right) \frac{\pi}{2} \right)}{\left( \left( \frac{j\varepsilon}{\varepsilon_1} \right)^2 - 1 \right)^2} (1 + \mathcal{O}(j^{-2}))
 \end{aligned}$$

In particular, when  $\varepsilon \rightarrow 0$ , then  $\tau_2$  tends to  $\Gamma_2 := \frac{2\sqrt{2}}{\pi} \left( \int_0^\infty \frac{x \sin^2((x-1)\pi/2)}{(x^2-1)^2} dx \right)^{\frac{1}{2}}$ , and we deduce from (9.8) and (9.11), plus the fact that  $\text{Im } \sqrt{\beta_j} = \mathcal{O}(e^{-\delta/\varepsilon})$  uniformly,

$$(9.13) \quad |A_{1,+} - A_{1,-}| \leq \tilde{\tau}_2 \left( \frac{2\varepsilon}{\pi} \sum_{j \geq 1} \text{Re } \sqrt{\beta_j} |b_j|^2 \right)^{\frac{1}{2}} + C e^{-(\pi L + \delta)/2\varepsilon},$$

where  $\tilde{\tau}_2$  can be taken arbitrarily close to  $\Gamma_2$ . Actually,  $\Gamma_2$  can be computed exactly, and one finds,

$$\Gamma_2 = \frac{2\sqrt{2}}{\pi} \left( -\frac{1}{2} + \frac{\pi}{4} \text{Si}(\pi) \right)^{\frac{1}{2}} \approx 0,879.$$

(Here,  $\text{Si}(x) := \int_0^x \frac{\sin t}{t} dt$ .)

Summing (9.10) with (9.13), and using the triangle inequality, we finally obtain

$$(9.14) \quad 2|A_{1,-}| \leq \tau_1 \sqrt{\alpha |A_{1,-}|^2 - \beta X + r_2(\varepsilon)} + \tau_2 \sqrt{X} + 3C e^{-(\pi L + \delta)/2\varepsilon},$$

where we have set

$$X := \frac{2\varepsilon}{\pi} \sum_j \text{Re } \sqrt{\beta_j} |b_j|^2.$$

Now, an elementary computation shows that the map

$$[0, A^2] \ni Y \mapsto \tau_1 \sqrt{A^2 - \beta Y^2} + \tau_2 Y$$

reaches its maximum at  $Y = \frac{\tau_2^2}{\beta \tau_1^2 + \tau_2^2} A / \sqrt{\beta}$ , and the maximum value is

$$\left( \sqrt{\tau_1^2 + \beta^{-1} \tau_2^2} \right) A.$$

Therefore, we deduce from (9.14),

$$\begin{aligned}
 (9.15) \quad 2|A_{1,-}| &\leq \left( \sqrt{\tau_1^2 + \beta^{-1} \tau_2^2} \right) \sqrt{\alpha |A_{1,-}|^2 + r_2(\varepsilon)} + 3C e^{-(\pi L + \delta)/2\varepsilon} \\
 &\leq \left( \sqrt{\alpha(\tau_1^2 + \beta^{-1} \tau_2^2)} \right) |A_{1,-}| + \mathcal{O}(e^{-(\pi L + \delta)/2\varepsilon}).
 \end{aligned}$$

Since  $\sqrt{\alpha(\tau_1^2 + \beta^{-1} \tau_2^2)}$  tends to  $\sqrt{\sum_{k \geq 2} k^{-3} + \Gamma_2^2}$  as  $\varepsilon \rightarrow 0$ , and

$$\sum_{k \geq 3} k^{-3} + \Gamma_2^2 \leq \frac{1}{4} + \frac{8}{10} < 4,$$

we have proved,

**Proposition 9.1.** *Under the assumption (5.1), there exist two constants  $C, \delta > 0$  such that, for any  $\varepsilon > 0$  small enough, one has,*

$$(9.16) \quad |A_{1,-}| \leq C e^{-(\pi L + \delta)/2\varepsilon}.$$

## 10. END OF THE PROOF

By Assumption **(H)**, we see that the Dirichlet eigenfunction  $u_0$  satisfies the hypothesis of [BHM] Lemma 3.1. Then, following the arguments of [BHM] leading to (13) in that paper, and using again [HM], Proposition 3.1 and Formula (5.13), we conclude that for any  $\delta > 0$  and any  $x \in (0, L)$ , there exists  $C_1$  such that the resonant state  $u_\varepsilon$  verifies (see [BHM], Formula (13)),

$$(10.1) \quad \|u_\varepsilon\|_{L^2([x, L] \times [-\varepsilon, \varepsilon])} \geq \frac{1}{C_0} \varepsilon^{4.5 + \delta} e^{-\pi x/2\varepsilon}.$$

Using this estimate, we can now prove as in [MN], Proposition 8.2, the following proposition, that contradicts the inequality (9.16), and thus completes the proof the theorem 2.1.

**Proposition 10.1.** *For any  $\delta > 0$ , there exists  $C > 0$ , such that*

$$(10.2) \quad |A_{1,-}| \geq \frac{1}{C} \varepsilon^{4.5 + \delta} e^{-\pi L/2\varepsilon},$$

for  $\varepsilon > 0$  small enough.

*Proof.* Starting from (9.5), we see,

$$(10.3) \quad \sum_{k \geq 1} |A_{k,+}|^2 \leq (1 + C e^{-\delta/\varepsilon}) |A_{1,-}|^2 + C e^{-(\pi L + \delta)/\varepsilon}.$$

Then, computing the quantity  $\|u_\varepsilon\|_{L^2([x, L] \times [-\varepsilon, \varepsilon])}$  by using the expression (8.1), we obtain (see [MN], proof of Proposition 8.2),

$$(10.4) \quad \begin{aligned} \|u_\varepsilon\|_{L^2([x, L] \times [-\varepsilon, \varepsilon])}^2 &\leq 4 \sum_{k \geq 1} |A_{k,+}|^2 + 4 \sum_{k > 1} |a_{k,-}|^2 e^{-2x \operatorname{Re} \theta_k / \varepsilon} \\ &\quad + \varepsilon |a_{1,-}|^2 e^{-2x \operatorname{Re} \theta_1 / \varepsilon}. \end{aligned}$$

Using (10.3) and (9.3), we deduce

$$(10.5) \quad \begin{aligned} \|u_\varepsilon\|_{L^2([x, L] \times [-\varepsilon, \varepsilon])}^2 &\leq C \varepsilon |a_{1,-}|^2 e^{-2x \operatorname{Re} \theta_1 / \varepsilon} + C |a_{1,-}|^2 e^{-2L \operatorname{Re} \theta_1 / \varepsilon} \\ &\quad + C \varepsilon^{-C} e^{-2x \operatorname{Re} \theta_1 / \varepsilon} e^{-2C_0 x / \varepsilon} + C e^{-(\pi L + \delta)/\varepsilon}, \end{aligned}$$

and thus, using (10.1), we finally obtain,

$$(10.6) \quad \varepsilon^{9+2\delta} \leq C |a_{1,-}|^2,$$

and the result is proved. □

## 11. AN EXTENSION TO LARGER DIMENSIONS

Here, we consider the similar problem in dimension  $n \geq 3$ , obtained by taking tubes with square sections. That is,  $\mathcal{C}$  is a regular bounded open subset of  $\mathbb{R}^n$ , and we have (in Euclidean coordinates  $x = (x_1, \dots, x_n) = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ ),

$$(11.1) \quad \begin{aligned} \bar{\mathcal{C}} &\subset \mathcal{B}; \\ (0, 0) &\in \partial\mathcal{C}; (L, 0) \in \partial\mathcal{B}; \\ [0, L] \times \{0\} &\subset \bar{\mathcal{B}} \setminus \mathcal{C}; \\ \{L\} \times Q_{\varepsilon_0} &\subset \partial\mathcal{B}, \end{aligned}$$

where  $Q_1 := \{(x_1, \dots, x_n); |x_j| < 1, j = 1, \dots, n\}$ , and  $Q_\varepsilon := \varepsilon Q_1$ . Then, setting  $\mathcal{T}(\varepsilon) := [-\varepsilon_0, L] \times Q_\varepsilon \cap (\mathbb{R}^n \setminus \mathcal{C})$ , and  $\mathbf{E} := \mathbb{R}^n \setminus \bar{\mathcal{B}}$ , we consider the resonances of the resonator  $\Omega(\varepsilon) := \mathcal{C} \cup \mathcal{T}(\varepsilon) \cup \mathbf{E}$ .

As before, let  $\lambda_0$  be an eigenvalue of  $-\Delta_{\mathcal{C}}$ , and let  $u_0$  be the corresponding normalized eigenfunction.

In this situation, the lower estimate of [HM] (see also [BHM]) becomes

$$\operatorname{Im} \rho(\varepsilon) = \mathcal{O}(e^{-(1-\delta)\pi L\sqrt{n-1}/\varepsilon}),$$

where  $\rho(\varepsilon)$  stands for any resonance that tends to  $\lambda_0$  as  $\varepsilon \rightarrow 0_+$ , and  $\delta > 0$  is arbitrary.

We assume again,

Assumption **(H)**:

$$\begin{aligned} \lambda_0 &\text{ is simple;} \\ u_0 &\text{ does not vanish on } \mathcal{C} \text{ near the point } (0, 0). \end{aligned}$$

Then, we have

**Theorem 11.1.** *Under Assume **(H)** and  $2 \leq n \leq 12$ . Then, for any  $\delta > 0$  there exists  $C_\delta > 0$  such that, the only resonance  $\rho(\varepsilon)$  close to  $\lambda_0$  satisfies,*

$$|\operatorname{Im} \rho(\varepsilon)| \geq \frac{1}{C_\delta} e^{-\pi(1+\delta)L\sqrt{n-1}/\varepsilon},$$

uniformly as  $\varepsilon \rightarrow 0_+$ .

*Proof.* The computations are very similar to those in dimension 2, and we highlight here only what is specific to dimension  $n$ . The notations are similar, but their meaning is modified as follows. For  $k = (k_2, \dots, k_n) \in \mathbb{N}^{n-1}$  (where  $\mathbb{N} := \{1, 2, 3, \dots\}$ ), we set

$$\begin{aligned} \alpha_k &:= \left( \frac{k_2\pi}{2}, \dots, \frac{k_n\pi}{2} \right) \in \mathbb{R}^{n-1}; \\ \theta_k &:= \sqrt{|\alpha_k|^2 - \varepsilon^2 \rho(\varepsilon)}; \\ \beta_k &:= |\alpha_k|^2 \varepsilon_1^{-2} - \rho(\varepsilon); \\ \psi_k(x') &:= \psi_{k_2}(x_2) \dots \psi_{k_n}(x_n); \\ \varphi_k(x') &:= \varphi_{k_2} \dots (x_2) \varphi_{k_n}(x_n). \end{aligned}$$

(Here,  $|k|$  stands for the Euclidean norm of  $k$  in  $\mathbb{R}^{n-1}$ .) With these notations, the formulas (8.1)-(8.6) remain valid with the following changes:

- $\sum_{k=1}^{\infty}$  must be replaced by  $\sum_{k \in \mathbb{N}^{n-1}}$ , and analog for  $\sum_{j=1}^{\infty}$ ;
- $y$  must be replaced by  $x'$ ;
- $(-\varepsilon, \varepsilon)$  and  $(-\varepsilon_1, \varepsilon_1)$  must be respectively replaced by  $Q_\varepsilon$  and  $Q_{\varepsilon_1}$  (where  $\varepsilon_1$  is taken such that  $\frac{(n-1)\pi^2}{4\varepsilon_1^2} > \lambda_0$ ).

Computing in two ways the quantities  $\langle v, \partial_x v \rangle_{\{L\} \times Q_1}$ ,  $\langle v, \varphi_{1,\dots,1} \rangle_{\{L\} \times Q_1}$ , and  $\langle \partial_x v, \psi_{1,\dots,1} \rangle_{\{L\} \times Q_\varepsilon}$ , we find the following analogs of (9.5)-(9.8):

$$\begin{aligned} & \sum_{k \in \mathbb{N}^{n-1}} (|k| - C|k|^{-1}e^{-\delta/\varepsilon})|A_{k,+}|^2 \\ & \leq (1 + Ce^{-\delta/\varepsilon})|A_{1,\dots,1,-}|^2 - \frac{2\varepsilon}{\pi}(1 + r_1) \sum_{j \in \mathbb{N}^{n-1}} \operatorname{Re} \sqrt{\beta_j} |b_j|^2 + r_2; \\ & |A_{1,\dots,1,+} + A_{1,\dots,1,-}| \leq Ce^{-(\pi L \sqrt{n-1} + \delta)/2\varepsilon} + \sum_{|k| > \sqrt{n-1}} \left| \frac{\mu_k}{\mu_{1,1}} A_{k,+} \right| \\ & \quad + \frac{C}{\sqrt{\varepsilon}} e^{-\pi L \sqrt{4+(n-2)^2}/2\varepsilon}; \\ & |A_{1,1,+} - A_{1,1,-}| \leq \frac{\varepsilon}{|\theta_{1,\dots,1}|} \sum_{j \in \mathbb{N}^{n-1}} |\nu_j \sqrt{\beta_j} b_j| + Ce^{-(\pi L \sqrt{n-1} + \delta)/2\varepsilon}, \end{aligned}$$

where we have set

$$\nu_j := \nu_{j_2} \dots \nu_{j_n} \quad ; \quad \mu_k := \mu_{k_2} \dots \mu_{k_n},$$

and with,

$$r_1 = \mathcal{O}(e^{-\delta/\varepsilon}) \quad ; \quad r_2 = \mathcal{O}(e^{-(\pi L \sqrt{n-1} + \delta)/\varepsilon}).$$

Using the fact that  $\mu_{k_2,\dots,k_n}/\mu_{1,\dots,1} \leq (k_2 - \frac{\varepsilon^2}{\varepsilon_1^2})^{-1} \dots (k_n - \frac{\varepsilon^2}{\varepsilon_1^2})^{-1}$  ( $k_2, \dots, k_n$  odd), this also gives

$$\begin{aligned} & |A_{1,\dots,1,+} + A_{1,\dots,1,-}| \\ (11.2) \quad & \leq \tilde{\tau}_1 \left( |A_{1,\dots,1,-}|^2 - \frac{\varepsilon}{\varepsilon_1} \sum_{j \in \mathbb{N}^{n-1}} |j| |b_j|^2 \right)^{\frac{1}{2}} + Ce^{-(\pi L \sqrt{2} + \delta)/2\varepsilon}, \end{aligned}$$

where  $\tilde{\tau}_1$  can be taken arbitrarily close to

$$(11.3) \quad J_1 := \left( \sum_{|k|^2 > n-1; k_j \text{ odd}} |k|^{-1} k_2^{-2} \dots k_n^{-2} \right)^{\frac{1}{2}} = \left( \sum_{k_j \text{ odd}} |k|^{-1} k_2^{-2} \dots k_n^{-2} - \frac{1}{\sqrt{n-1}} \right)^{\frac{1}{2}}.$$

A rough estimate on  $J_1$  can be obtained by writing,

$$J_1^2 \leq \frac{1}{\sqrt{n-1}} \left( \left( \sum_{\ell \in \mathbb{N}^{\text{odd}}} \frac{1}{\ell^2} \right)^{n-1} - 1 \right) \leq \frac{1}{\sqrt{n-1}} \left( \left( \frac{\pi^2}{8} \right)^{n-1} - 1 \right).$$

In a similar way we obtain,

$$(11.4) \quad |A_{1,\dots,1,+} - A_{1,\dots,1,-}| \leq \tilde{\tau}_2 \left( \frac{\varepsilon}{\varepsilon_1} \sum_{j \in \mathbb{N}^{n-1}} |j| |b_j|^2 \right)^{\frac{1}{2}} + C e^{-\pi L(\sqrt{n-1}+\delta)/2\varepsilon},$$

where  $\tilde{\tau}_2$  can be taken arbitrarily close to the quantity

$$(11.5) \quad J_2 = \frac{4^{n-1}}{(\pi\sqrt{2})^{n-1}\sqrt{n-1}} \left( \int_{\mathbb{R}_+^{n-1}} \frac{|x| \sin^2((x_1-1)\pi/2) \dots \sin^2((x_{n-1}-1)\pi/2)}{(x_1^2-1)^2 \dots (x_{n-1}^2-1)^2} dx_1 \dots dx_{n-1} \right)^{\frac{1}{2}}.$$

Writing  $|x| \leq |x_1| + \dots + |x_{n-1}|$  and making permutations on the variables, we obtain,

$$J_2 \leq \frac{4^{n-1}}{(\pi\sqrt{2})^{n-1}} \left( \int_0^{+\infty} \frac{t \sin^2((t-1)\pi/2)}{(t^2-1)^2} dt \right)^{\frac{1}{2}} \left( \int_0^{+\infty} \frac{\sin^2((t-1)\pi/2)}{(t^2-1)^2} dt \right)^{\frac{n-2}{2}}$$

Setting

$$L_1 := \int_0^{+\infty} \frac{t \sin^2((t-1)\pi/2)}{(t^2-1)^2} dt \quad ; \quad L_2 := \int_0^{+\infty} \frac{\sin^2((t-1)\pi/2)}{(t^2-1)^2} dt,$$

it becomes,

$$J_2 \leq \left( \frac{L_1}{L_2} \right)^{\frac{1}{2}} \left( \frac{4\sqrt{L_2}}{\pi\sqrt{2}} \right)^{n-1}.$$

The integrals  $L_1$  and  $L_2$  can be computed exactly, and one finds,

$$L_1 = -\frac{1}{2} + \frac{\pi}{4} \text{Si}(\pi) \approx 0.9545 \quad ; \quad L_2 = \frac{\pi^2}{8}.$$

In particular, for  $\varepsilon$  small enough, we have

$$(11.6) \quad \tilde{\tau}_1^2 + \tilde{\tau}_2^2 < \frac{8}{10} + \frac{1}{\sqrt{n-1}} \left( \left( \frac{\pi^2}{8} \right)^{n-1} - 1 \right),$$

and one can check that this quantity is strictly less than 4 when  $2 \leq n \leq 12$ .

At this point, we can complete the proof as in the 2 dimensional case.  $\square$

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